The University of Chicago

A NEW THEORY OF PARAMETRIC PROBLEMS IN THE CALCULUS OF VARIATIONS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
FRANK LYNWOOD WREN

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This dissertation is one of a group of five submitted during the year 1930 to the Department of Mathematics of the University of Chicago and concerned with topics in the theory of the Calculus of Variations. The group as a whole has been published in book form by the University of Chicago Press under the title *Contributions to the Calculus of Variations, 1930.*

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A New Theory of Parametric Problems in the Calculus of Variations.

Introduction. Among the different methods so far given for deducing Jacobi's necessary condition in the calculus of variations the one which seems to be best adapted to the discussion of parametric problems is that given by G.A.Bliss in a paper published in April, 1916.[1] The integral studied by Professor Bliss is

$$J = \int f(y_1, \cdots, y_n; y_1', \cdots, y_n')\, dt = \int f(y, y')\, dt$$

and the arcs along which it is taken have equations of the form

$$y_i = y_i(t), \qquad (t_1 \leq t \leq t_2).$$

He attains the Jacobi condition by means of a theory of the arcs

$$\eta_i = \eta_i(t), \qquad (t_1 \leq t \leq t_2),$$

in (t, η)-space which minimize the second variation of the integral J and which have the further property that they are normal variations, that is, they must satisfy the relation

$$\sum_{i=1}^{n} y_i' \eta_i = 0$$

The purpose of this paper is the discussion of the Jacobi condition and the Hamilton-Jacobi theory for parametric problems in a form which is different from that studied by Bliss, and which for certain problems, notably the general theory of relativity, is more convenient. The integral to be

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1. G.A.Bliss, "Jacobi's condition for problems of the calculus of variations in parametric form". Trans. of the Amer. Math. Soc., vol. 17 (1916), pp 195-206.

considered is

$$I = \int [2f(x^1, x^2, \cdots\cdots, x^n; x^{1'}, x^{2'}, \cdots\cdots, x^{n'})]^{\frac{1}{2}} dt = \int [2f(x, x')]^{\frac{1}{2}} dt$$

where $f(x, x')$ is homogeneous of degree two in the variables x' and positive at all points of a certain region R.

The outline of the first five sections below is, in general, similar to that of the paper by Professor Bliss. After stating the problem and deriving certain fundamental formulae in Section 1 we proceed, in the next three sections, to study the first and second variations and the necessary conditions for a minimum which can be derived therefrom. In the study of the second variation it is found desirable to use orthogonal variations, as defined in Section 3, instead of the normal variations used by Professor Bliss. In Section 5 certain properties of the analogues of the Jacobi equations are discussed and a mechanism for determining the conjugate points is set up.

In Section 6, which contains a discussion of the Hamilton-Jacobi theory as related to the integral I, the Hamilton function is defined, and in terms of this function the canonical differential equations are set up. There is also given the construction of the function $W(x)$, defining the transversal sub-spaces for a field F, which satisfies the Hamilton-Jacobi partial differential equation. By means of an n-parameter family of such functions $W(x)$ the extremals for the original problem can be secured.

1. Notations and formulae. The integral to be minimized has the form

$$(1.1) \qquad I = \int_{t_1}^{t_2} [2f(x,x')]^{1/2}\,dt$$

where x and x' are to denote the sets $(x^1, x^2, \text{----}, x^n)$, $(x^{1'}, x^{2'}, \text{-------}, x^{n'})$ respectively, and where the primes denote differentiation with respect to the parameter t.

Let R be a region of 2n-dimensional points (x,x') with $x' \neq 0$, having the property that if (x,x') is interior to R so are all points (x,kx') for $k>0$. In order to carry through the analysis of the problem, it will be assumed that $f(x,x')>0$ and of class C^{IV}[2], at least, in R. Furthermore, in order to insure the invariance on the integral I under a change of parametric representation, it will be assumed that f(x,x') satisfies the homogeneity relation

$$(1.2) \qquad f(x,kx') = k^2 f(x,x') \quad , \quad (k>0).$$

The integral is to be taken along arcs with equations of the form

$$x^i = x^i(t) \quad , \quad (t_1 \leq t \leq t_2).$$

Such an arc will be called admissible if it is continuous and consists of a finite number of sub-arcs on each of which the derivatives $x^{i'}(t)$ are continuous, and if, furthermore, all of its elements (x,x') are interior to R.

The statement of the problem is then as follows:

In the sub-class of admissible arcs joining two fixed points 1 and 2, to find one which furnishes a minimum value of the integral I.

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2. The class of a function defined as in O. Bolza, "Vorlesungen über Variationsrechnung",(1909), p 13.

As a consequence of the homogeneity condition (1-2) we have the following relations:[3]

$$(1.3)\quad \begin{aligned} &f_{\alpha}x^{\alpha'}=2f\ , \quad f_{i\alpha}x^{\alpha'}=f_i\ , \quad f_{\alpha x^i}x^{\alpha'}=2f_{x^i}\ ,\\ &f_{\alpha\beta}x^{\alpha'}x^{\beta'}=2f\ , \quad f_{i\alpha x^k}=f_{ix^k}\ , \end{aligned}$$

where Greek letters indicate a summation, as is customary in tensor analysis, and where the following notations are used:

$$f_i=\frac{\partial f}{\partial x^{i'}}\ , \qquad f_{x^i}=\frac{\partial f}{\partial x^i}\ ,$$

$$f_{ix^j}=\frac{\partial^2 f}{\partial x^{i'}\partial x^j}\ , \quad f_{ijx^k}=\frac{\partial^3 f}{\partial x^{i'}\partial x^{j'}\partial x^k}\ .$$

2. The first variation. In order to construct the first and second variations of the integral I let us consider the one-parameter family of curves

$$(2.1)\quad x^i(t)=X^i(t)+a\eta^i(t)\ , \quad (t_1\leq t\leq t_2\ ;\ i=1,2,\cdots,n),$$

where $\eta^i(t)$ represents a set of admissible variations defined by the conditions[4]

$$\eta^i(t) \text{ of class } D' \text{ for } t_1\leq t\leq t_2\ ,$$

$$\eta^i(t_1)=\eta^i(t_2)=0\ .$$

The value of the integral (1·1) for the family of curves (2·1) is

$$I(a)=\int_{t_1}^{t_2}\left[2f(x+a\eta\ ,\ x'+a\eta')\right]^{1/2}dt\ .$$

If the value of I(0) is to be a minimum along an arc E_{12} it is necessary that I'(0)=0 and I"(0)≧0 for all admissible variations η^i. From the above integral we obtain the following derivatives

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3. J.H.Taylor, "Reduction of Euler's equations to a canonical form". Bull. of the Amer. Soc.; vol. 31 (1925) pp 257-262.

4. See Bolza, loc.cit., p 63.

$$(2.2)\qquad I'(0)=\int_{t_1}^{t_2}\frac{1}{(2f)^{1/2}}\left[f_{x^\alpha}\eta^\alpha+f_{\dot{x}^\alpha}\eta^{\alpha'}\right]^{1/2}dt\ ,$$

$$I''(0)=\int_{t_1}^{t_2}\left\{\frac{1}{(2f)^{1/2}}\left[f_{x^\alpha x^\beta}\eta^\alpha\eta^\beta+2f_{x^\alpha\dot{x}^\beta}\eta^{\alpha'}\eta^\beta+f_{\dot{x}^\alpha\dot{x}^\beta}\eta^{\alpha'}\eta^{\beta'}\right]-\frac{1}{(2f)^{3/2}}\left[f_{x^\alpha}\eta^\alpha+f_{\dot{x}^\alpha}\eta^{\alpha'}\right]^2\right\}dt,$$

where the arguments of f and its derivatives are $x^i(t)$, $x^{i'}(t)$.

From the first variation I'(0) it follows by the method of Dubois-Reymond[5] that there exists a set of constants c_i such that the equations

$$f_i/(2f)^{1/2}=\int_{t_1}^{t}f_{x^i}/(2f)^{1/2}\,dt+c_i$$

hold at every point of the minimizing arc E_{12}, even if it has corner points. From these equations we readily deduce

Theorem I. On each sub-arc of a minimizing arc the functions $f_i/(2f)^{1/2}$ are of class C' between corners and the Euler differential equations

$$(2.3)\qquad \frac{d}{dt}\,f_i/(2f)^{1/2}-f_{x^i}/(2f)^{1/2}=0$$

are satisfied. If the parameter t is so chosen that f=constant along the minimizing arc then these equations reduce to the system

$$(2.4)\qquad \frac{d}{dt}f_i-f_{x^i}=0\ .$$

Theorem II. Along a minimizing arc the functions $f_i/(2f)^{1/2}$ must be continuous, even at corners, and if the parameter t is so selected that f = constant along the arc then the functions f_i are continuous.

We have so far made no assumption concerning the exis-

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5. Bolza, loc. cit., p 27.

tence of a second derivative $x^{i\prime\prime}(t)$ along the minimizing arc E_{12}. If the parameter t is so chosen that $f(x,x')$ is constant along E_{12} we obtain by means of the application of the existence theorem for implicit functions[6] to the Dubois-Reymond equations:

Theorem III. Near a point between corners of a minimizing arc E_{12}, at which the determinant $|f_{ij}|$ is different from zero, the functions $x^i(t)$, defining E_{12}, are of class C^{IV} at least, provided the parameter t is so selected that $f(x,x')$ is constant along E_{12}.

When the functions $x^i(t)$ have continuous derivatives of the second order we can carry out the differentiation indicated in (2·3) and obtain

$$P_i = \frac{d}{dt} f_{x^i}/(2f)^{1/2} - f_{x^i}/(2f)^{1/2} = \frac{1}{(2f)^{1/2}}\left(f_{i\alpha}x^{\alpha\prime\prime} + f_{i\alpha\beta}x^{\alpha\prime}x^{\beta\prime} - f_{x^i}\right) - f_i f'/(2f)^{3/2} = 0.$$

By application of (1·3) these equations become

$$(2\cdot5)\qquad P_i = \frac{1}{(2f)^{1/2}}\left[f_{i\alpha}x^{\alpha\prime\prime} + \frac{1}{2}\left(f_{i\alpha x^\beta} + f_{i\beta x^\alpha} - f_{\alpha\beta x^i}\right)x^{\alpha\prime}x^{\beta\prime}\right] - f_i f'/(2f)^{3/2} = 0.$$

If we now assume that the matrix (f_{ij}) is of rank n and denote by f^{ij} the element of the reciprocal matrix corresponding to the element f_{ij} so that

$$f^{i\alpha}f_{\alpha k} = f_{i\alpha}f^{\alpha k} = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k, \end{cases}$$

the equations (2·5) may be written in a much simpler form. For this purpose we employ the Christoffel symbols $[ij,k]$ and $\{ij,k\}$, which have the following definitions and properties:

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6. Goursat-Hedrick, "Mathematical Analysis" (1904), vol. I, p 45; also G.A.Bliss, "Princeton Colloquium Lectures", Colloquium Series of Amer. Soc.,vol. III (1913)

$$(2\cdot6)\qquad \begin{aligned} [ij,k] &= \tfrac{1}{2}\left(f_{ikx^j}+f_{jkx^i}-f_{ijx^k}\right), \\ \{ij,k\} &= f^{k\alpha}[ij,\alpha] \quad , \quad [ij,k]=f_{k\alpha}\{ij,\alpha\}. \end{aligned}$$

Equations (2·5) may now be written as follows:[7]

$$(2\cdot7)\qquad P_i = f_{i\alpha}Q^\alpha/(2f)^{1/2} - f_i f'/(2f)^{3/2} = 0,$$

where

$$Q^i = x^{i\prime\prime} + \{\alpha\beta, i\}\, x^{\alpha\prime}x^{\beta\prime}.$$

Lemma I. The expressions P_i and Q^i satisfy the relations

$$f' = f_\alpha Q^\alpha \quad , \quad P_i = \frac{1}{(2f)^{3/2}}\left(2f f_{i\alpha} - f_i f_\alpha\right) Q^\alpha.$$

To prove this we note that

$$f' = f_{x^r}x^{r\prime} + f_{x^r}x^{r\prime\prime}.$$

From (1·3) and (2·6) we obtain

$$\begin{aligned} f_{x^r}x^{r\prime} &= \tfrac{1}{2} f_{\alpha\beta x^r}\, x^{\alpha\prime}x^{\beta\prime}x^{r\prime} \\ &= \tfrac{1}{2}\left(f_{\alpha\gamma x^\beta} + f_{\beta\gamma x^\alpha} - f_{\alpha\beta x^r}\right) x^{\alpha\prime}x^{\beta\prime}x^{r\prime} \\ &= f_\gamma \{\alpha\beta,\gamma\}\, x^{\alpha\prime}x^{\beta\prime}. \end{aligned}$$

Hence

$$f' = f_\gamma\left(x^{\gamma\prime\prime} + \{\alpha\beta,\gamma\}\, x^{\alpha\prime}x^{\beta\prime}\right) = f_\gamma Q^\gamma.$$

The second formula of the lemma follows immediately by substituting this value of f' in (2·7).

We therefore have the results that, along an arc satisfying the equations $Q^i = 0$, the function f is constant and $P_i = 0$. The solutions of $P_i = 0$ are called extremals. Hence we have

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7. These relations and others which follow in Lemma I, and Theorems IV and V were developed by Professor Bliss in his course entitled "Applications of the Calculus of Variations to Problems in Mathematical Physics", University of Chicago, Autumn Quarter, 1926. See also J.H.Taylor, loc. cit.

also the following theorem:

Theorem IV. Near every point where the matrix (f_{ij}) is of rank n the solution of the equations

$$(2\cdot8) \qquad \mathfrak{L}^i = x^{i''} + \left\{\alpha\beta, i\right\} x^{\alpha'} x^{\beta'} = 0$$

are all extremals along which the parameter t is so chosen that f is constant.

The equations (2·8) form a system of second order differential equations in the x's which are solvable for the $x^{i''}$. For such a system there is a unique solution through each set of initial values $(t,x,x') = (\tau,\xi,\xi')$ in a sufficiently small neighborhood of a particular extremal arc $E_{/2}$[8]. Each solution of (2·8) is thus imbedded in a 2n-parameter family of such solutions defined by functions

$$x^i = x^i(t,\xi,\xi')$$

which, for t = 0, take with their derivatives the initial values $\xi^i, \xi^{i'}$. Since the equations (2·8) do not contain t explicitly, it follows that the functions

$$x^i = x^i(t-\tau, \xi, \xi')$$

are also solutions for arbitrary values of τ. These are the solutions of equations (2·8), which for $t=\tau$, instead of t = 0, take with their derivatives the initial values $\xi^i, \xi^{i'}$.

Theorem V. For every extremal arc $E_{/2}$ the parameter t can be so chosen that f = constant along the arc. If, furthermore, the determinant $|f_{ij}|$ is different from zero along $E_{/2}$,

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8. G.A.Bliss, "The solutions of differential equations as functions of their constants of integration". Bull. of Amer. Math. Soc., vol. 25 (1918) p 15.

then the functions $x^i(t)$ defining the extremal must be solutions of equations (2·8).

To prove this we note first that the equations $P_i = 0$ are not independent, since, by (1·3), we have

$$x^{\alpha'} P_\alpha = x^{\alpha'} \frac{d}{dt} \frac{f_\alpha}{(2f)^{1/2}} - \frac{f_{x^\alpha} x^{\alpha'}}{(2f)^{1/2}}$$

$$= \frac{d}{dt} \frac{f_\alpha x^{\alpha'}}{(2f)^{1/2}} - \frac{f_\alpha x^{\alpha''} + f_{x^\alpha} x^{\alpha'}}{(2f)^{1/2}}$$

$$= \frac{d}{dt} (2f)^{1/2} - \frac{d}{dt} (2f)^{1/2} = 0.$$

Hence it is permissible, in order to determine solutions, to adjoin some arbitrary equation which, with $P_i = 0$, is solvable for the $x^{i''}$. We may, in particular, so select the parameter that f = constant along E_{12}. Then we have the system $P_i = 0$, $f' = 0$ or, with the help of Lemma I, the equivalent system

$$(2f f_{i\alpha} - f_i f_\alpha) Q^\alpha = 0 \quad , \quad f_\alpha Q^\alpha = 0 \, ,$$

whose matrix of coefficients of the expressions Q^i has the same rank as the matrix

$$\begin{pmatrix} f_{ij} \\ f_i \end{pmatrix}$$

of dimensions n by n + 1.

Hence, near each point where the matrix (f_{ij}) is of rank n, we have $Q^i = 0$ as a consequence of $P_i = 0$, $f' = 0$.

If S be any sub-space

$$\xi^i = \xi^i(u_1, u_2, \cdots, u_{n-1}) \, , \quad (i = 1, 2, \cdots, n)$$

which is cut by a certain extremal arc E_{12} but does not have E_{12} tangent to it, then every extremal arc in a neighborhood, at least, of E_{12} will also cut S. Consequently the point of intersection of each extremal with S can be taken as the ini-

tial point ξ^i of the extremal for t = 0. Furthermore, the initial values $\xi^{i'}$ can be subjected to the condition f = 1 without losing any extremals. The initial values $\xi^i, \xi^{i'}$ thus characterized form a (2n-2)-parameter family, and we have therefore a (2n-2)-parameter family of extremals for the original problem.

3. The Weierstrass E-function and Legendre's Quadratic Form. The character of the integrand function assumed for the integral (1·1) is such that the expressions for the Weierstrass E-function and Legendre's quadratic form are somewhat different from those usually given. It is, therefore, desirable to set up these expressions for the function $[2f(x,x')]^{\frac{1}{2}}$.

By means of (1·3) the Weierstrass E-function is found to be

$$E(x,x',X') = [2f(x,X')]^{\frac{1}{2}} - \frac{f_\alpha(x,x')X^{\alpha'}}{[2f(x,x')]^{\frac{1}{2}}},$$

and Legendre's quadratic form is

$$Q(x,x',\xi) = \frac{f_{\alpha\beta}(x,x')\xi^\alpha\xi^\beta}{[2f(x,x')]^{\frac{1}{2}}} - \frac{[f_\alpha(x,x')\xi^\alpha]^2}{[2f(x,x')]^{\frac{3}{2}}}.$$

These expressions must satisfy the necessary conditions of Weierstrass and Legendre,[9] respectively.

Let us now define an orthogonal variation η^i as one which satisfies the condition[10] $f_{\alpha\beta}\eta^\alpha x^{\beta'} = 0$.
For all orthogonal variations the Legendre necessary condition

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9. G.A.Bliss, "Calculus of Variations", p 131.
10. L.P.Eisenhart, "Riemannian Geometry", (1926), p 38.

reduces to

$$f_{\alpha\beta}\,\eta^{\alpha}\eta^{\beta} \geq 0.$$

Lemma II. *If $f_{\alpha\beta}\eta^{\alpha}\eta^{\beta} > 0$ for every set of orthogonal variations η^{i}, then the determinant $|f_{ij}|$ is necessarily different from zero.*

For suppose $|f_{ij}| = 0$. Then there exists a set η^{i}, not all zero, such that $f_{i\alpha}\eta^{\alpha} = 0$. If we multiply these n equations by η^{i} and sum, then by $x^{i'}$ and sum, we should have $f_{\alpha\beta}\eta^{\alpha}\eta^{\beta} = 0$ and $f_{\alpha\beta}x^{\alpha'}\eta^{\beta} = 0$ satisfied simultaneously. Thus η^{i} would be, contrary to hypothesis, an orthogonal variation for which

$$f_{\alpha\beta}\eta^{\alpha}\eta^{\beta} = 0.$$

4. *The second variation.* It was established in Theorem V of Section 2 that for every extremal arc E_{12} the parameter t can be so chosen that f(x,x') is constant along the arc. It follows then, as a consequence of the homogeneity property of f(x,x'), that we can secure 2f(x,x') = 1 along an extremal. We shall, therefore, in order to simplify the discussion of the second variation and the deduction of Jacobi's necessary condition, assume that the parameter has been so chosen that 2f(x,x') = 1 along the extremal arc E_{12} to be considered.

Under the above hypothesis the expression (2·2), found for the second variation, may, by means of (2·4), be reduced to

$$(4\cdot1)\qquad I''(0) = \int_{t_1}^{t_2} \left\{ 2\omega(t,\eta,\eta') - (f_{\alpha}\eta^{\alpha})'^{2} \right\} dt,$$

where

$$(4\cdot2)\qquad 2\omega(t,\eta,\eta') = f_{x^{\alpha}x^{\beta}}\eta^{\alpha}\eta^{\beta} + 2f_{\alpha x^{\beta}}\eta^{\alpha'}\eta^{\beta} + f_{\alpha\beta}\eta^{\alpha'}\eta^{\beta'},$$

and the arguments of f and its derivatives are the functions

$x^i(t)$, $x^{i'}(t)$ belonging to the extremal arc E_{12}. From (4·2) we derive the following formulae:

$$(4\cdot3)\quad \omega_{\eta i} = f_{x^i x^\alpha}\eta^\alpha + f_{\alpha x^i}\eta^{\alpha'}\ ,\quad \omega_{\eta i'} = f_{i x^\alpha}\eta^\alpha + f_{i\alpha}\eta^{\alpha'}\ ,$$

$$(4\cdot4)\quad 2\omega = \eta^\alpha \omega_{\eta\alpha} + \eta^{\alpha'}\omega_{\eta\alpha'}\ ,\quad \eta^\alpha \omega_{\beta\alpha} + \eta^{\alpha'}\omega_{\beta\alpha'} = f^\alpha \omega_{\eta\alpha} + f^{\alpha'}\omega_{\eta\alpha'}\ .$$

The condition $I''(0) \geq 0$ suggests a new problem in the calculus of variations, namely that of minimizing I"(0) in the class of arcs of the (t,η)-space joining the fixed points $(t,\eta)=(t_1,0)$, $(t,\eta)=(t_2,0)$ [11].

The Euler equations for this problem are

$$(4\cdot5)\quad J_i(\eta) = \frac{d}{dt}\omega_{\eta i'} - \omega_{\eta i} - (f_\alpha \eta^\alpha)'' f_i = 0\ .$$

These equations will be called the Jacobi equations of the original problem in the (t,x)-space.

From (4·3), (1·3), and (2·4) we have as an identity in $\eta^{i'}$

$$\begin{aligned} x^{\alpha'}\left(\frac{d}{dt}\omega_{\eta\alpha'} - \omega_{\eta\alpha}\right) &= x^{\alpha'}\left\{\frac{d}{dt}\left(f_{\alpha x^\beta}\eta^\beta + f_{\alpha\beta}\eta^{\beta'}\right) - \left(f_{x^\alpha x^\beta}\eta^\beta + f_{x^\alpha\beta}\eta^{\beta'}\right)\right\} \\ &= \frac{d}{dt}\left(2f'_\alpha\eta^\alpha + f_\alpha\eta^{\alpha'}\right) - f''_\alpha\eta^\alpha - f'_\alpha\eta^{\alpha'} \\ &= (f_\alpha\eta^\alpha)''. \end{aligned}$$

Furthermore, for $\eta^i = (ct+d)x^{i'}$, we obtain by (4·3), (2·4), and (1·3)

$$\begin{aligned} \frac{d}{dt}\omega_{\eta i'} - \omega_{\eta i} &= \frac{d}{dt}\left\{(ct+d)\left[f_{ix^\alpha}x^{\alpha'} + f_{i\alpha}x^{\alpha''}\right] + c f_{i\alpha}x^{\alpha'}\right\} - (ct+d)\left(f_{x^i x^\alpha}x^{\alpha'} + f_{\alpha x^i}x^{\alpha''}\right) - c f_{\alpha x^i}x^{\alpha'} \\ &= \frac{d}{dt}\left[(ct+d)\,f_{x^i} + c f_i\right] - \left[(ct+d)\frac{d}{dt}f_{x^i} + 2c f_{x^i}\right] \\ &= \frac{d}{dt}\left[(ct+d)\,f_{x^i} + c f_i\right] - \frac{d}{dt}\left[(ct+d)f_{x^i} + c f_i\right] = 0. \end{aligned}$$

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11. G.A.Bliss, "Jacobi's condition for problems of the calculus of variations in parametric form". Trans. of Amer. Math. Soc. vol 17 (1916) p 199.

Hence we have

Theorem VI. Every orthogonal solution of $J_i(\eta) = 0$ satisfies

$$(4\cdot6)\quad K_i(\eta) = \frac{d}{dt}\omega_{\eta i'} - \omega_{\eta i} = 0\ ,$$

and every solution of $K_i(\eta)=0$ makes $(f_\alpha \eta^\alpha)'' = 0$; hence every solution of $K_i(\eta)=0$, with initial values at t_1 making $f_\alpha \eta^\alpha = (f_\alpha \eta^\alpha)' = 0$ at t_1, is an orthogonal solution of $J_i(\eta)=0$. Further, $\eta^i(t)=(ct+d)x^{i'}$ is a solution of $K_i(\eta)=0$ for every c and d.

The Jacobi condition for the arc E_{12} is to be found under the assumption that E_{12} is a minimizing arc which has no corners and along which $f_{\alpha\beta}\eta^\alpha\eta^\beta > 0$ for every orthogonal variation η^i. Then, by Lemma II, the determinant $|f_{ij}|$ is necessarily different from zero along E_{12} and, by Theorem III, the arc E_{12} is of class C^{IV} at least.

Lemma III. If a solution of the equations $K_i(\eta)=0$ vanishes with its derivatives at a particular value of t, then the solution is identically zero.

To prove this lemma we note first that the equations $K_i(\eta)=0$, which may by (4·3) be written in the form

$$\frac{d}{dt}\left(f_{i\alpha}\eta^{\alpha'} + f_{ix^\alpha}\eta^\alpha\right) - \left(f_{x^i x^\alpha}\eta^\alpha + f_{\alpha x^i}\eta^{\alpha'}\right) = 0\ ,$$

are linear and of the second order in the η^i. The determinant of the coefficients of the $\eta^{i''}$ is the determinant $|f_{ij}|$ which is different from zero. Hence the equations may be put into normal form by solving for the $\eta^{i''}$. From the existence theorems for such differential equations we know that, if the initial

values of $\eta^i, \eta^{i'}$ are such that $\eta^i = \eta^{i'} = 0$, the only solution of [12] $K_i(\eta) = 0$ that corresponds to these initial values is $\eta^i \equiv 0$.

Since all orthogonal solutions of $J_i(\eta) = 0$ are, by Theorem VI, also orthogonal solutions of $K_i(\eta) = 0$, and since for such orthogonal solutions I"(0) reduces to

$$I''(0) = \int_{t_1}^{t_2} 2\,\omega(t, \eta, \eta')\, dt$$

[13]

we now have the further theorem:

Theorem VII. If E_{12} is a minimizing arc without corners along which $f_{\alpha\beta}\gamma^\alpha\gamma^\beta > 0$ for every orthogonal variation η^i, then no orthogonal solution $u^i(t) \not\equiv 0$ of the Jacobi equations $J_i(\eta) = 0$ for E_{12} can exist satisfying the relations $u_i(t_1) = u_i(t_3) = 0$ for a value t_3 between t_1 and t_2.

Definition. A point 3 is said to be conjugate to 1 on the arc E_{12} if there exists an orthogonal solution $u^i(t) \not\equiv 0$ of the Jacobi equations $J_i(\eta) = 0$ for E_{12} such that $u^i(t_1) = u^i(t_3) = 0$.

The Jacobi condition, which is equivalent to the result stated in Theorem VII, may now be formulated as follows:

Jacobi's Necessary Condition: If E_{12} is a minimizing arc without corners along which $f_{\alpha\beta}\gamma^\alpha\gamma^\beta > 0$ for every orthogonal variation η^i, then no point 3 conjugate to 1 can lie between 1 and 2 on the arc E_{12}.

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12. G.A.Bliss, "Solutions of differential equations of the first order as functions of their initial values", Annals of Math., 2nd series, vol. 6 (Jan. 1905) p 49.
13. The method of proving this theorem, with the exception that orthogonal solutions are used instead of normal solutions, is the same as that used by Bliss in his paper of references 1 and 11.

5. Properties of the solutions of the equations $J_i(\eta)=0$, $K_i(\eta)=0$ and the determination of the conjugate points. In the previous section certain relations were seen to exist between the solutions of the equations $J_i(\eta)=0$ and those of $K_i(\eta)=0$. We shall now proceed to further examine the solutions of these two systems of equations and, by means of the properties derived, to set up a mechanism for determining the points 3 conjugate to 1 on the extremal arc E_{12}. It will, at all times, be understood that the arguments of f and its derivatives are the $x^i(t)$, $x^{i\prime}(t)$ belonging to the minimizing arc E_{12} which is without corners and along which $f_{\alpha\beta}\eta^\alpha\eta^\beta>0$ for every orthogonal variation η^i. Furthermore it will be assumed that the parameter t has been so chosen that $2f(x,x')=1$ along E_{12}.

Theorem VIII. The Jacobi equations satisfy the relation $x^{\alpha\prime}J_\alpha(\eta)=0$ identically in $t, \eta^i, \eta^{i\prime}, \eta^{i\prime\prime}$ and hence are dependent.

For from (1·3) and (2·4) applied to (4·3) when $\eta^i=x^{i\prime}$, $\eta^{i\prime}=x^{i\prime\prime}$ we obtain

$$\omega_{x^{i\prime}}=f_{x^ix^\alpha}x^{\alpha\prime}+f_{\alpha x^i}x^{\alpha\prime\prime}=f_{x^i}',$$

$$\omega_{x^{i\prime\prime}}=f_{ix^\alpha}x^{\alpha\prime}+f_{i\alpha}x^{\alpha\prime\prime}=f_i', \tag{5·1}$$

$$x^{\alpha\prime}\omega_{\eta^{\alpha\prime}}=f_{\alpha x^\beta}x^{\alpha\prime}\eta^\beta+f_{\alpha\beta}x^{\alpha\prime}\eta^{\beta\prime}=f_{x^\alpha}\eta^\alpha+(f_\alpha\eta^\alpha)'.$$

and hence by (1·3),(4·4),(5·1) and (2·4) we have

$$\begin{aligned}x^{\alpha\prime}J_\alpha(\eta)&=x^{\alpha\prime}\frac{d}{dt}\omega_{\eta^{\alpha\prime}}-x^{\alpha\prime}\omega_{\eta\alpha}-(f_\alpha\eta^\alpha)''\\&=\frac{d}{dt}x^{\alpha\prime}\omega_{\eta^{\alpha\prime}}-(\eta^{\alpha\prime}\omega_{x^{\alpha\prime\prime}}+\eta^\alpha\omega_{x^{\alpha\prime}})-(f_\alpha\eta^\alpha)''\\&=\frac{d}{dt}x^{\alpha\prime}\omega_{\eta^{\alpha\prime}}-\left[\eta^{\alpha\prime}f_{x^\alpha}+\eta^\alpha f_{x^\alpha}'+(f_\alpha\eta^\alpha)''\right]\end{aligned}$$

Theorem IX. To every solution η^i of the equations $K_i(\eta)=0$ there corresponds a unique orthogonal solution of the form $\eta^i-(ct+d)x^{i'}$.

For, the two equations

$$0=f_\alpha[\eta^\alpha-(ct+d)x^{\alpha'}]=f_\alpha\eta^\alpha-(ct+d),$$

$$0=(f_\alpha\eta^\alpha)'-c$$

determine c and d for a fixed value of t, and hence uniquely determine the function ct+d. We know by Theorem VI that, for this value of ct+d, the difference $\eta^i-(ct+d)x^{i'}$ is a solution of $K_i(\eta)=0$ since these equations are linear in the η^i. Furthermore this solution, by Theorem VI, is an orthogonal solution.

Corollary. To every solution η^i of the equations $K_i(\eta)=0$ there corresponds a unique orthogonal solution, of the form $\eta^i-(ct+d)x^{i'}$, of the Jacobi equations $J_i(\eta)=0$.

The existence theorems for differential equations[14] provide us with a 2n-parameter family of solutions of the equations $K_i(\eta)=0$ and hence, by Theorem VI, with a (2n-2)-parameter family of orthogonal solutions of the Jacobi equations $J_i(\eta)=0$. Let (u^{ik}), (v^{ik}) be matrices of n rows and n-1 columns in which the columns are solutions of the equations

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14. G.A.Bliss, "The solutions of differential equations as functions of their initial values." Annals of Math. 2nd series, Vol. 6(1905) p 49.

$K_i(\eta)=0$, and let $(u^{ik'})$, $(v^{ik'})$ be matrices whose elements are the derivatives of the respective elements of (u^{ik}) and (v^{ik}). It follows then that the determinant

$$\Delta(t)=\begin{vmatrix} u^{ik} & v^{ik} & x^{i'} & tx^{i'} \\ u^{ik'} & v^{ik'} & x^{i''} & x^{i'}+tx^{i''} \\ u^{ik} & v^{ik} & x^{i'} & 0 \\ u^{ik'} & v^{ik'} & x^{i''} & x^{i'} \end{vmatrix}$$

is either identically zero or else everywhere different from zero, and in the latter case the solutions (u^{ik}), (v^{ik}), $x^{i'}$, $tx^{i'}$ are called a fundamental system of solutions of the equations $K_i(\eta)=0$.[15] Each column of the matrices (u^{ik}), (v^{ik}) of D(t) can be made into an orthogonal solution of the equations $J_i(\eta)=0$ by choosing constants c, d and multiplying the next to the last column by (ct + d) and the last column by c and subtracting the results from the column in question. Hence we may define a fundamental system of orthogonal solutions of the equations $J_i(\eta)=0$ to be a system of orthogonal solutions represented by the matrices (u^{ik}), (v^{ik}) altered as just described above, for which the determinant D(t) is everywhere different from zero.

Lemma IV. A necessary and sufficient condition that the 2n-2 orthogonal solutions (u^{ik}), (v^{ik}) of $J_i(\eta)=0$ form a fundamental system is that they be linearly independent.

For, suppose that the matrices of orthogonal solutions (u^{ik}), (v^{ik}) form a fundamental system. Then no constants

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15. E. Goursat, "Cours d'Analyse Mathematique", Tome II (1905) pp 413-419.

k_i, l_i, not all zero, exist such that $k_\alpha u^{i\alpha}(t) + l_\alpha v^{i\alpha}(t) = 0$ for every value of t. Otherwise we should have this solution vanishing with its derivative at some particular value of t and, by Theorem VI and Lemma III, identically zero. The same would then be true of D(t). Hence we have the result that the 2n - 2 orthogonal solutions of a fundamental system are linearly independent.

Conversely, suppose (u^{ik}), (v^{ik}) matrices of linearly independent orthogonal solutions of the equations $J_i(\eta) = 0$. If at some particular value of t, say $t = t_1$, $D(t_1) = 0$, constants k_i, l_i, K, λ, not all zero, can be determined such that

$$k_\alpha u^{i\alpha}(t_1) + l_\alpha v^{i\alpha}(t_1) + K x^{i'}(t_1) = 0,$$
$$k_\alpha u^{i\alpha'}(t_1) + l_\alpha v^{i\alpha'}(t_1) + K x^{i''}(t_1) + \lambda x^{i'}(t_1) = 0.$$

If we multiply the first n of these equations by $f_i'(x(t_1), x^{i'}(t_1))$ and sum we have that $K = 0$. Similarly multiplying the first equations by f_i' and the second by f_i and adding we find that $\lambda = 0$, thus leading to a contradiction of our assumption that the u^{ik}, v^{ik} are linearly independent. Hence D(t) must be everywhere different from zero and the matrices (u^{ik}), (v^{ik}) of orthogonal solutions form a fundamental system.

Lemma V. <u>Every orthogonal solution $u^i(t)$ of the equations $J_i(\eta) = 0$ can be expressed in terms of a fundamental system of 2n-2 orthogonal solutions (u^{ik}), (v^{ik}), in the form $u^i(t) = k_\alpha u^{i\alpha}(t) + l_\alpha v^{i\alpha}(t)$.</u>

For the equations

$$u^i(t_1) = k_\alpha u^{i\alpha}(t_1) + l_\alpha v^{i\alpha}(t_1) + K x^{i'}(t_1),$$
$$u^{i'}(t_1) = k_\alpha u^{i\alpha'}(t_1) + l_\alpha v^{i\alpha'}(t_1) + K x^{i''}(t_1) + \lambda x^{i'}(t_1),$$

can be solved for the constants $k_\alpha, \epsilon_\alpha, K, \lambda$ and, as above, since $u^i(t)$ is an orthogonal solution, $K=\lambda=0$. Hence we have the orthogonal solution $u^i(t) - k_\alpha u^{i\alpha}(t) - \epsilon_\alpha u^{i\alpha}(t)$ vanishing with its derivative at $t=t_1$ and, by Theorem VI and Lemma III, it is identically zero.

Lemma VI. *If (u^{ik}) is a matrix of n rows and n-1 columns of which the latter are linearly independent orthogonal solutions of the equations $f_i(y)=0$ with elements vanishing at t_1, then every orthogonal solution $u^i(t)$ of $f_i(y)=0$, with elements vanishing at t_1, is expressible in the form $u^i(t) = k_\alpha u^{i\alpha}(t)$.*

Under the hypothesis of the lemma the matrix $(u^{ik'})$ is of rank n-1 at t_1. Otherwise constants k_i, not all zero, would exist such that $k_\alpha u^{i\alpha'}(t_1) = 0$, in which case the orthogonal solution $k_\alpha u^{i\alpha}(t)$ would be identically zero, and this would imply that the columns of (u^{ik}) were not linearly independent.

Since $u^i(t)$ and $(u^{ik}(t))$ are orthogonal solutions we have

$$f_\alpha u^{\alpha i}(t) = 0 \quad , \quad f'_\alpha u^{\alpha i}(t) + f_\alpha u^{\alpha i'}(t) = 0 \, ,$$
$$f_\alpha u^{\alpha}(t) = 0 \quad , \quad f'_\alpha u^{\alpha}(t) + f_\alpha u^{\alpha'}(t) = 0 \, .$$

Since at t_1 we have $u^{ik}(t_1)=0, u^i(t_1)=0$, it follows from the derived equations above that $f_\alpha u^{\alpha i}=0$ and $f_\alpha u^{\alpha'}=0$ at $t = t_1$. Consequently, from (1-3) and the fact that $|f_{ij}|$ is different from zero, we have that the determinant $|u^{i'} u^{ik'}|$ vanishes at $t = t_1$. Hence the equations $u^{i'}(t_1) - k_\alpha u^{i\alpha'}(t_1) = 0$ can be solved for the

constants k_i and we have the result that the orthogonal solution $u^i(t) - k_\alpha u^{i\alpha}(t)$, which vanishes with its derivative at $t = t_1$, is by Theorem VI and Lemma III identically zero.

Denote by $\Delta(t, t_1)$ the determinant

$$\Delta(t, t_1) = \begin{vmatrix} u^{ik}(t) & v^{ik}(t) & x^{i'}(t) & 0 \\ u^{ik}(t_1) & v^{ik}(t_1) & 0 & x^{i'}(t_1) \end{vmatrix}$$

and note that each column of (u^{ik}), (v^{ik}) may be made into an orthogonal solution of $J_2(\eta) = 0$ by multiplying the last two columns of $\Delta(t, t_1)$ by a suitably determined factor $(ct+d)$ and subtracting the results from the column in question. We then have[16]

Theorem X. *If $\Delta(t, t_1)$ is not identically zero, then its zeros determine the points 3 conjugate to 1 on the extremal arc E_{12}.*

Theorem XI. *If the matrix (u^{ik}) is such that the determinant $|u^{ik} x^{i'}|$ has rank one at t, but is not identically zero, then the conjugate points to 1 on the extremal arc E_{12} are determined by the zeros of the determinant.*

The existence theorems for differential equations applied to (2·4) justify the existence of a (2n-2)-parameter family of linearly independent solutions of the type

$$x^i = \phi^i(t; a^1, a^2, \cdots, a^{n-1}; b^1, b^2, \cdots, b^{n-1}).$$

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16. The method of proof for these two theorems is similar to that used by Bliss in his paper of references 1 and 11. The only difference is that here one uses orthogonal solutions instead of the normal solutions, and the multipliers here are $f_i(x(t), x'(t))$ instead of $y'(t)$.

Upon substituting these solutions in (2·4) we obtain an identity which, upon differentiating with respect to the parameters a^k and b^k, exhibits $(\varphi_{a^k}^{i})$, $(\varphi_{b^k}^{i})$ as matrices of solutions of the equations $K_i(\eta) = 0$ of the type specified in Theorem X.

In a similar manner[17] we have the existence of a matrix $(\varphi_{a^k}^{i})$ of solutions of the type described in Theorem XI. It is obtained from the solution

$$x^i = \varphi^i(t; a^1, a^2, \cdots\cdots, a^{n-1})$$

of the equations (2·4), all of which arcs pass through the point 1.

It can also be shown that there is a first conjugate point $t_3 > t_1$[17].

6. The Hamilton-Jacobi Theory. We shall in this section define the Hamiltonian function for the original integral (1·1) and, in terms of this function, set up the canonical differential equations the solutions of which provide us with solutions of the Euler differential equations. Furthermore, by means of the Hilbert invariant integral belonging F with slope functions $p^i(x)$, we shall construct the function $W(x)$ which satisfies the Hamilton-Jacobi partial differential equation and defines the sub-spaces W(x) = constant transversal to F. By means of a complete integral $W(x; a^1, \cdots, a^{n-1})$ of the Hamiltonian equation we can determine all of the ex-

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17. G.A.Bliss, "Jacobi's condition for problems of the calculus of variations in parametric form". Trans. of Amer. Math. Soc. vol 17 (1916) pp 205-206.

tremals for the original problem.

It will be assumed that the minimizing arc E_{12} has no corners and that it is an arc along which $f_{\alpha\beta}\eta^\alpha\eta^\beta > 0$ for every orthogonal variation η^i. Furthermore, it will be assumed that the parameter t has been so chosen that $2f(x,x')=1$ along E_{12}.

We now introduce the Canonical Variables x^i, u_i where the u_i are defined for the region R by the equations

$$u_i = f_i(x,x') . \tag{6.1}$$

These equations, when the determinant $|f_{ij}|$ is different from zero, are solvable for the variables $x^{i'}$ as functions of the x^i and u_i in the form

$$x^{i'} = P^i(x,u) . \tag{6.2}$$

Furthermore, from the homogeneity property of $f_i(x, x')$ it is clear that the equations (6·1) with u_i replaced by $k u_i$, for $k>0$, have the solution $x^{i'} = k P^i(x,u)$, from which we obtain the property

$$P^i(x,ku) = k P^i(x,u) .$$

In terms of the canonical variables the Hamilton Function will be defined to be

$$H(x,u) = [2f(x,P)]^{1/2} - 1 . \tag{6.3}$$

As a consequence of this definition and the identities

$$u_i = f_i(x,P) \tag{6.4}$$

we have

Lemma VII. The derivatives of the Hamiltonian function $H(x,u)$ have the values

$$(6\cdot5)\qquad H_{x^i} = \frac{-f_{x^i}(x,P)}{[2f(x,P)]^{1/2}}, \qquad H_{u_i} = \frac{P^i(x,u)}{[2f(x,P)]^{1/2}}$$

For, if we differentiate (6·4) first with respect to x^j and then with respect to u_j we obtain

$$(6\cdot6)\qquad 0 = f_{ix^j} + f_{i\alpha}\frac{\partial P^\alpha}{\partial x^j}, \qquad \delta^i_j = f_{i\alpha}\frac{\partial P^\alpha}{\partial u_j},$$

where $\delta^i_j = 0$ for $i \neq j$, and $\delta^i_i = 1$.

If we now differentiate (6·3) first with respect to x^i, then with respect to u_i, and simplify by means of (1·3) and (6·6) we obtain

$$H_{x^i} = \frac{1}{(2f)^{1/2}}\left[f_{x^i} + f_\alpha\frac{\partial P^\alpha}{\partial x^i}\right] = \frac{1}{(2f)^{1/2}}\left[f_{x^i} + f_{\alpha\beta}P^\beta\frac{\partial P^\alpha}{\partial x^i}\right]$$

$$= \frac{1}{(2f)^{1/2}}\left[f_{x^i} - f_{\beta x^i}P^\beta\right] = -f_{x^i}/(2f)^{1/2},$$

$$H_{u_i} = f_\alpha\frac{\partial P^\alpha}{\partial u_i}/(2f)^{1/2} = f_{\alpha\beta}P^\beta\frac{\partial P^\alpha}{\partial u_i}/(2f)^{1/2}$$

$$= \delta^\beta_i P^\beta/(2f)^{1/2} = P^i/(2f)^{1/2},$$

where the arguments of f and its derivatives are x^i and P^i.

From the results obtained above we now derive

Theorem XII. Every solution $x^i(t)$ of the Euler differential equations in the form

$$(6\cdot7)\qquad \frac{d}{dt}f_i(x,x') - f_{x^i}(x,x') = 0, \qquad 2f(x,x') = 1$$

defines by means of equations (6·1), functions $u_i(t)$ which

together with the $x^i(t)$ satisfy the equations

$$(6\cdot8)\qquad x^{i\prime}(t) = H_{u_i}(x,u)\ ,\quad u_i'(t) = -H_{x^i}(x,u)\ ,$$
$$H(x,u) = [2f(x,p)]^{1/2} - 1 = 0\ ,$$

and conversely.

For, the functions $x^i(t)$ which are solutions of (6·7) define a set of functions $u_i(t)$ by means of (6·1). Then from (6·2), (6·5), and (6·7) it follows that equations (6·8) are satisfied.

Conversely, functions $x^i(t)$ and $u_i(t)$ which are solutions of (6·8) are such that $x^{i\prime} = p^i(x,u)$, and hence equations (6·4) are satisfied. It follows then from (6·4), the second equation (6·8), and (6·5) that the $x^i(t)$ are solutions of (6·7).

Lemma VIII. Along every solution of the equations

$$(6\cdot9)\qquad x^{i\prime}(t) = H_{u_i}(x,u)\ ,\quad u_i'(t) = -H_{x^i}(x,u)\ ,$$

the function $H(x,u)$ is constant.

For

$$\frac{dH}{dt} = H_{x^\alpha}x^{\alpha\prime} + H_{u_\alpha}u_\alpha' \equiv 0\ .$$

The equations (6·9) will be designated as the Canonical Differential Equations for the original problem. The application of the usual existence theorems for differential equations to these equations gives us solutions in the form

$$x^i = x^i(t-t_0, x_0^i, u_{i0})\ ,\quad u_i = u_i(t-t_0, x_0^i, u_{i0})\ ,$$

which for $t=t_0$ take the values $x^i=x^i_0$, $u_i = u_{i0}$. These solutions for $H(x,u)\equiv 0$ are, by Theorem XII, solutions of the Euler differential equations of the original problem. By the method of Section 2 the initial values x^i_0 may be expressed in terms of n-1 independent parameters, and the relation $H(x,u) = 0$ at $t = t_0$ imposes a further restriction on the u_{i0}. As a consequence of these remarks and the further fact that the parameter t_0 is not an essential parameter it follows that the solutions of the canonical differential equations provide us with a (2n-2)-parameter family of solutions of the Euler differential equations.

A field for the original integral (1·1) is defined to be a region E of x-space with which is associated a set of slope-functions $p^i(x)$ of class C' such that the Hilbert integral

$$I^* = \int \frac{f_\alpha(x,p)}{[2f(x,p)]^{1/2}}\, dx^\alpha$$

is independent of the path of integration. On account of the homogeneity property of f the above definition is equivalent to saying that a region F of the x-space is a field if there exists a set of slope-functions $p^i(x)$ satisfying the condition $2f(x,p) = 1$ and such that the integral

$$\int f_\alpha(x,p)\, dx^\alpha$$

is independent of the path of integration.

A curve through the point x with direction dx is said to intersect a second curve through x with direction x' transversally if the condition

$$(6\cdot10) \qquad f_\alpha(x,x')\,\delta x^\alpha = 0$$

is satisfied. A sub-space $W(x)$ =constant is said to cut a curve E transversally if every curve of the sub-space through the point of intersection of E with the sub-space cuts E transversally. For directions dx in the space $W(x)$ = constant we have by differentiation

$$W_{x^\alpha}\,\delta x^\alpha = 0,$$

and if (6·10) is to be a consequence of this relation we must have

$$W_{x^i} = \mu f_i(x,x').$$

Since the integral

$$I^* = \int f_\alpha(x,p)\,\delta x^\alpha$$

is independent of the path in a field with slope-functions $p^i(x)$ satisfying the condition 2f (x,p)=1 there must be a function $W(x)$ in the field for which[18]

$$(6\cdot11) \qquad W_{x^i} = f_i(x,p).$$

Furthermore the equations (6·11) show that the slope-functions $p^i(x)$ of the field are determined in terms of W_{x^i} by the equations

$$(6\cdot12) \qquad p^i = P^i(x, W_x).$$

Therefore we have

Theorem XIII. A necessary and sufficient condition that a region F be a field is that there exists in it slope-functions $p^i(x)$ and a function $W(x)$ such that the

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18. E.Goursat, "Cours d'Analyse Mathematique", Tome I(1902) pp 354-357.

equations

$$(6\cdot13) \qquad W_{x^i} = f_i(x,p) \quad , \quad 1 = 2f(x,p) \; ,$$

hold identically throughout F.

Only the converse of the theorem needs additional justification. If the equations (6·13) hold identically throughout F it follows that the equations (6·12) are satisfied and

$$I^* = \int_{x_0}^{x} f_\alpha(x,p)\,dx^\alpha = \int_{x_0}^{x} W_{x^\alpha}\,dx^\alpha$$
$$= W(x) - W(x_0)$$

so that I^* is independent of the path of integration. Consequently F is a field with the slope-functions $p^i = P^i(x, W_x)$.

Theorem XIV. For every field F with slope-functions $p^i(x)$ the function W(x), which defines the transversal subspace of the field, satisfies the Hamilton-Jacobi partial differential equation

$$(6\cdot14) \qquad H(x, W_x) = [2f(x, P(x, W_x))]^{1/2} - 1 = 0$$

identically in x^i throughout F. Conversely, every solution W (x) of equation (6·14) in a region F defines in F a field with the slope-functions $p^i = P^i(x, W_x)$.

For, by Theorem XIII, the function W(x), defining the transversal sub-space, is such that equations (6·13) are satisfied identically throughout F. From the first n of these equations we have $p^i = P^i(x, W_x)$. It then follows from the last equation of (6·13) that equation (6·14) is satisfied identically in the x^i throughout F.

Conversely, suppose $W(x)$ a solution of (6·14) in a region F. If we define slope-functions for the region F by the equations $p^i(x) = P^i(x, W_x)$ then, from the argument on page 26, $W_{x^i} = f_i(x,p)$ and hence, by Theorem XIII, the region F with the slope-functions $p^i(x)$ is a field.

Theorem XV. *If $W(x; a^1, a^2, \cdots, a^{n-1})$ is a solution of class C" of equation (6·14) with the matrix $(W_{x^i a^k})$ of rank n-1, then the equations*

$$W_{a^k}(x; a) = b^k \qquad (k = 1, 2, \cdots, n-1) \tag{6·15}$$

define a (2n-2)-parameter family of extremals for the original problem.

To prove this theorem let us suppose that $x^i(t)$ is a solution of (6·15) for fixed values of a^k, b^k. Then if we define

$$u_i(t) = W_{x^i}(x(t); a)$$

and differentiate with respect to t, we obtain

$$u_i'(t) = W_{x^i x^\alpha} x^{\alpha\prime}. \tag{6·16}$$

Now differentiate (6·15), with $x^i(t)$ substituted, with respect to t and get

$$W_{a^k x^\alpha} x^{\alpha\prime} = 0. \tag{6·17}$$

Since the function $W(x; a)$ satisfies (6·14) identically in x^i and a^k, we obtain the two following identities by differentiating (6·14) first with respect to x^i and then with respect to a^k

$$H_{x^i} + H_{u_\alpha} W_{x^\alpha x^i} = 0, \tag{6·18}$$

$$H_{u_\alpha} W_{x^\alpha a^k} = 0. \tag{6·19}$$

Since the matrix $(W_{x^i a^k})$ is of rank n-1 we have from (6·17) and (6·19) that

$$(6\cdot20) \qquad x^{i\prime}(t) = \mu(t)\, H_{u_i}(x,u) .$$

From equations (6·16), (6·20), and (6·18) we then obtain

$$(6\cdot21) \qquad u_i'(t) = -\mu(t)\, H_{x^i}(x,u) .$$

The equations (6·20) and (6·21) considered simultaneously are, except for the factor $\mu(t)$, the canonical differential equations (6·10) the solutions of which give us, for $H(x,u) \equiv 0$, a (2n-2)-parameter family of extremals of the original problem. By proper choice of the parameter the factor $\mu(t)$ can be made to equal unity. The extremals of the original problem remain extremals under parametric transformations that preserve direction and would consequently remain extremals, except possibly for sigh, under any parametric transformation. It therefore follows that equations (6·20) and (6·21) considered simultaneously give us, except possibly for sign, a (2n-2)-parameter family of extremals of the original problem.

Vita.

Frank Lynwood Wren was born in Yorkville, Gibson County, Tennessee on September 28, 1894. His elementary education was obtained in the public schools of Greenfield, Tennessee and his high school education in McFerrin School, Martin, Tennessee . Upon graduating from McFerrin School in 1911 he entered the University of the South, Sewanee, Tennessee from which institution he graduated with the A.B. (optime merens) degree in 1915. During his senior year there he served as student assistant in Mathematics and French, also as assistant proctor of the University. From 1915 to 1917 he was employed as principal and teacher of Mathematics and French in the City High School, Martin, Tennessee. In the fall of 1917 he began his graduate work in Mathematics at Johns Hopkins University, Baltimore, Maryland, also serving as student assistant in Mathematics. While there he had work with Professors Frank Morley and Abraham Cohen. This work was interrupted by enlistment in the United States army in December, 1917 which led into service as private. in Camp Meade, Maryland and, from July, 1918 to May, 1919, overseas as sergeant in charge of payrolls for the enlisted men of the 79th Division. During the school years 1919-20, 1920-21 he taught Mathematics and Modern Languages in McFerrin School, Martin, Tennessee, attending the University of Chicago during the summer quarter 1920.

He was married June 25, 1921 to Miss Alleyne Talley

and to this union two children have been born. During the summer of 1921 he again attended Johns Hopkins University for six weeks taking work with Professor Murnaghan.

In the fall of 1921 he began teaching at McCallie School, Chattanooga, Tennessee, where he remained until June, 1924, serving as Head of the Mathematics Department and Master in charge of Dormitories during the last two years of his stay there.

He attended George Peabody College for Teachers, Nashville, Tennessee for six quarters during the years 1923-25, receiving his M.A. degree in June, 1925. While at Peabody he was elected a member of Phi Delta Kappa, an honorary educational fraternity.

In the fall of 1925 he entered the University of Chicago and since that time has been in residence there eight quarters. He has taken courses with Professors Barnard, Bartky, Bliss, Coble, Dickson, Graustein, Lane, MacMillan, Moulton, and Slaught. His thesis was suggested by Professor Bliss and the study has been carried out under his careful and patient direction and criticism. This opportunity is taken to extend to Professor Bliss an expression of appreciation for his direction and criticism, without which the study could not have been completed.

Since the fall of 1927 he has been teaching in George Peabody College for Teachers as Associate Professor of the Teaching of Mathematics.